LOCAL MOVE FORMULAE FOR THE ALEXANDER POLYNOMIALS OF n-KNOTS

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ABSTRACT. It is well-known:Suppose there are three 1-dimensional links K_+, K_-, K_0 such that K_+, K_-, and K_0 coincide out of a 3-ball B trivially embedded in S^3 and that $K_+ \cap B$, $K_- \cap B$, and $K_0 \cap B$ are drawn as follows. Then $\Delta_{K_+} - \Delta_{K_+} = (t-1) \cdot \Delta_{K_0}$, where Δ_K is the Alexander polynomial of K.

Figure1

We know similar formulae of other invariants of 1-dimensional knots and links. (The Jones polynomial etc.)

It is natural to ask: Suppose there are two n-dimensional knots K_+, K_- and a submanifold K_0 such that K_+, K_-, and K_0 coincide out of a n-ball B trivially embedded in S^{n+2}. Then is there a relation in $K_+ \cap B$, $K_- \cap B$, and $K_0 \cap B$ with the following property(*)? (*)If K_+, K_-, and K_0 satisfy this relation, an invariant of K_+, that of K_-, and that of K_0 satisfy a fixed relation.

In this paper we pove there are such a relation where K_+, K_-, and K_0 satisfy the formula $\Delta_{K_+} - \Delta_{K_+} = (t-1) \cdot \Delta_{K_0}$, where Δ_K is a polynomial to represent the Alexander polynomial of K.

We show another relation where K_+, K_-, and K_0 satisfy the formula
$\mathrm{Arf} K_+ - \mathrm{Arf} K_- = \{|bP_{4k+2} \cap I(K_0)| + 1\} mod 2$,
where (1)$I(\quad)$ is the inertia group. and $I(K_0)$ is the inertia group of a smooth manifold which is orientation preserving diffeomorphic to K_0. (2)For a group G, $|G|$ denote the order of G.

A local move formula is a relation of an invariant of a few knots related by a local move as above.

1. Introduction

It is well-known:Suppose there are three 1-dimensional links K_+, K_-, K_0 such that K_+, K_-, and K_0 coincide out of a 3-ball B trivially embedded in S^3 and that $K_+ \cap B$,

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$K_- \cap B$, and $K_0 \cap B$ are drawn as follows. Then $\Delta_{K_+} - \Delta_{K_+} = (t-1) \cdot \Delta_{K_0}$, where Δ_K is the Alexander polynomial of K.

Figure1

We know similar formulae of other invariants of 1-dimensional knots and links. (The Jones polynomial etc. See §5 of [11]. See also [7] [10].)

It is natural to ask: Suppose there are two n-dimensional knots K_+, K_- and a submanifold K_0 such that K_+, K_-, and K_0 coincide out of a n-ball B trivially embedded in S^{n+2}. Then is there a relation in $K_+ \cap B$, $K_- \cap B$, and $K_0 \cap B$ with the following property(*)? (*)If K_+, K_-, and K_0 satisfy this relation, an invariant of K_+, that of K_-, and that of K_0 satisfy a fixed relation.

In this paper we pove there are such a relation where K_+, K_-, and K_0 satisfy the formula $\Delta_{K_+} - \Delta_{K_+} = (t-1) \cdot \Delta_{K_0}$, where Δ_K is a polynomial to represent the Alexander polynomial of K.

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where (1)$I(\quad)$ is the inertia group. and $I(K_0)$ is the inertia group of a smooth manifold which is orientation preserving diffeomorphic to K_0. (2)For a group G, $|G|$ denote the order of G.

A *local move formula* is a relation of an invariant of a few knots related by a local move as above.

[17] is a preprint of this paper.

The author proved another local move formulae in [18][19].

2. Review of the Alexander polynomials for n-knots and n-links

We review the Alexander polynomials for n-knots and n-links and n-submanifolds. See [2] [15], [16], [20] for detail.

We work in the smooth category. Let $K = (K_1, ..., K_m)$ be an n-dimensional closed oriented submanifold of S^{n+2}. It is known any tubular neighborhood of K is $K \times D^2$. (See P.49, 50 of [14].) Put $X = \overline{S^{n+2} - K \times D^2}$. Then any S^1 in X is oriented by using the orientation of S^{n+2} and that of K. Let $\iota : S^1 \to X$ denote the embedding. Take a homomorphism $\alpha : H_1(X;\mathbb{Z}) \to \mathbb{Z}$ such that $\alpha \circ \iota_* : H_1(S^1;\mathbb{Z}) \to \mathbb{Z}$ carries $+1$ to $+1$. Then the infinite cyclic covering $\pi : \widetilde{X} \to X$ associated with α is called the *cannonical cyclic covering* of K.

We can regard $H_p(\widetilde{X};\mathbb{Z})$ as a $\mathbb{Z}[t,t^{-1}]$-module by using the covering translation $\widetilde{X} \to \widetilde{X}$ defined by α.

We can also regard $H_p(\widetilde{X};\mathbb{Q})$ as a $\mathbb{Q}[t,t^{-1}]$-module.

Module theory says that any $\mathbb{Q}[t,t^{-1}]$-module is congruent to

$(\mathbb{Q}[t,t^{-1}]/\lambda_1) \oplus ... \oplus (\mathbb{Q}[t,t^{-1}]/\lambda_l) \oplus (\oplus^k \mathbb{Q}[t,t^{-1}])$, where $\lambda_* \in \mathbb{Q}[t,t^{-1}]$ is not zero and λ_* is not the $\mathbb{Q}[t,t^{-1}]$-balanced class of 1.
Two polynomials $f(t), g(t) \in \mathbb{Q}[t,t^{-1}]$ are said to be $\mathbb{Q}[t,t^{-1}]$-*balanced* (written $f \doteq g$) if there is an integer n and a nonzero rational number r such that $f(t) = r \cdot t^n \cdot g(t)$.
Let $H_p(\widetilde{X};\mathbb{Q})$ be $\mathbb{Q}[t,t^{-1}]/\lambda_1 \oplus ... \oplus \mathbb{Q}[t,t^{-1}]/\lambda_l \oplus^k \mathbb{Q}[t,t^{-1}]$ as above. *The* $\mathbb{Q}[t,t^{-1}]$-*p-Alexander polynomial* is the $\mathbb{Q}[t,t^{-1}]$-balanced class of the product $\lambda_1 \cdot ... \cdot \lambda_l$ if $k = 0$, where k is the rank of the free part. *The* $\mathbb{Q}[t,t^{-1}]$-*p-Alexander polynomial* is 0 if $k \neq 0$. If $H_p(\widetilde{X};\mathbb{Q}) \cong 0$, *The* $\mathbb{Q}[t,t^{-1}]$-*p-Alexander polynomial* is 1.

We discuss the $\mathbb{Q}[t,t^{-1}]$ module case but our results can be extended to the $\mathbb{Z}[t,t^{-1}]$ module case.

In this paper we mainly discuss the case where K is a knot although we discuss other cases a little. Furthermore, our results can be extended to some other cases without heavy difficulty.

If K above is a connected smooth manifold which is PL homeomorphic to the standard sphere, K is called *n-(dimensional)knot* (See [6] etc).

3. Main results

In this section, we prove local move formulae for n-knots $\subset S^{n+2}$. ($n \geqq 3$.)

Let K_+, K_- be an n-knot $\subset S^{n+2}$ ($n \geqq 3$). Let K_0 be an n submanifold $\subset S^{n+2}$. Let B be an $(n+2)$-ball trivially embedded in S^{n+2}. Suppose that K_+ coincides with K_- in $S^{n+2} - B$. Note that there is a Seifert hypersurface V_+ (resp. V_-) for K_+ (resp. K_-) such that V_+ coincides with V_- in $S^{n+2} - B$. Suppose that $V_+ \cap B$ (resp. $V_- \cap B$) is a disjoint union of an $(n+1)$-dimensional p-handle h^p_+ (resp. h^p_-) and an $(n+1)$-dimensional $(n+1-p)$-handle h^{n+1-p}_+ (resp. h^{n+1-p}_-) which are attached to ∂B and which are embedded trivially in B. Let $p \neq n+1-p$. Suppose that h^{n+1-p}_+ coincides with h^{n+1-p}_-. Suppose that the linking number (in B) of '$h^p_+ \cup (-h^p_-)$' and 'h^{n+1-p}_+ whose attached part is fixed in ∂B' is one if an orientation is given. The concept is drawn in Figure 2. Let K_0 be $\partial(V_+ - \mathrm{Int}B)$. [Figure 2]
Then we say that (K_+, K_-, K_0) is related by the *$(p, n+1-p)$-move.*

We draw the figure of the $(1,2)$-move case (the case if $p = 1$ and $n = 2$) in Note below Theorem 4.1.

Let $n = 4k+1$ in the above case. Suppose that K_+, K_-, V_+, V_- satisfy the same condition at $S^{4k+3} - B$ as in (i). Suppose $V_+ \cap B$ (resp. $V_- \cap B$) is a $(4k+2)$-dimensional $(2k+1)$-handle h_+ (resp. h_-). Supoose that the core of h_+ (resp. h_-) is trivially embedded in B. Push off the core in the positive direction of the normal bundle of V_+ (resp. V_-) in B. Note that we can consider the framing (in B) of h_+ (resp. h_-) . Suppose that the framing of h_+ (resp. h_-) is 0 (resp. 1) if an orientation is given. Let K_0 be $\partial(V_+ - \mathrm{Int}B)$. (The 1-dimensional case of this relation among K_+, K_-, K_0 is one in Figure 1.)

Then we say that $(K_+,\, K_-,\, K_0)$ is related by the *$XXII$-move.*

Note 3.1. One way of saying, when we make K_-, K_0, we just operate in B and we do not need the diffeomorphism type or the hmeomorphism type of K_-, K_0. In this meaning, we use the word 'local' in the above definition.

Theorem 3.2. *Let K_+, K_- be n-knots $\subset S^{n+2}$ $(n \geqq 3)$. Let K_0 be n-submanifold $\subset S^{n+2}$. Suppose that $(K_+,\, K_-,\, K_0)$ is related by the $(p, n+1-p)$-move. Then we have:*
$\Delta^p_{K_+} - \Delta^p_{K_+} = (t-1)\cdot\Delta^p_{K_0}$,
where Δ^p_K is a polynomial whose balanced class is the p-Alexander polynomial for K.

Theorem 3.3. *Let K_+, K_- be $(4k+1)$-knots. Let K_0 be a closed oriented $(4k+1)$-submanifold $\subset S^{4k+3}$. Suppose that $(K_+,\, K_-,\, K_0)$ is related by the XXII-move. Then we have:*
$\Delta^{(2k+1)}_{K_+}(t) - \Delta^{(2k+1)}_{K_-}(t) = (t-1)\cdot\Delta^{(2k+1)}_{K_0}(t)$.
where $\Delta^{(2k+1)}_K$ is a polynomial whose balanced class is the p-Alexander polynomial for K.

Note. For $(4k+3)$-knots, we can define XXII-move. However, note the following: Suppose $K_-$and K_0 satisfy the XXIIrelation for a $(4k+3)$-knot K_+. Then K_0 is not a knot in general. Because: there is an example such that $K_+ \cong S^3$ and $K_- \cong \mathbb{R}P^3$.

Theorem 3.4. *Let K_+, K_-, K_0, B be as in Theorem 3.3. Let bP_{4k+2} be the bP-subgroup $\subset \Theta^{4k+1}$. Suppose bP_{4k+2} is not congruent to the trivial group. Then we have*
$\mathrm{Arf}K_+ - \mathrm{Arf}K_- = \{|bP_{4k+2} \cap I(K_0)| + 1\}mod2$,
where (1)$I(\quad)$ is the inertia group. $I(K_0)$ is the inertia group of a smooth manifold which is orientation preserving diffeomorphic to K_0.
(2)For a group G, $|G|$ denote the order of G.

Note. See [KM] for the bP-subgroup. See [Kk] [BS] for the inertia group.
Proof of Theorem 3.2 and 3.3. Let V_* be a compact oriented $(n+1)$-submanifold $\subset S^{n+2}$ such that $\partial V_* = K_*$ (their orientaion are compatible). Recall V_* is called a Seifert hypersurface for K_*. In Theorem 3.3 we put $n = 4k+1$ and $p = 2k+1$.
Take X_* $\widetilde{X}_*$ as in §2. Let $Y_* = X - V_* \times [-1,1]$, $V_* \times [-1,1]$ is the tubular neibourhood of V_* in Y. Consider the Meyer-Vietoris exact sequence:
$H_*(\amalg^{\infty}_{-\infty} V_*) \xrightarrow{f_*} H_*(\amalg^{\infty}_{-\infty} Y_*) \to H_p(\widetilde{X}_*;\mathbb{Z})$.
There are V_* such that f_* is represented by the following matrixes:

$$P_+ = \{p^+_{ij}\}, P_- = \{p^-_{ij}\}, P_0 = \{p^0_{ij}\},$$

such that (1) P_+ and P_- are $(n+1)\times(n+1)$ matrices. P_0 is an $n\times n$ matrix. (2)$p^+_{n+1,n+1} - p^-_{n+1,n+1} = t-1$, (3)$p^+_{ij} = p^-_{ij} = p^0_{ij}$. $(1 \leqq i \leqq n,\, 1 \leqq j \leqq n,)$.
By calculus of determinants, $\det P_+ - \det P_- = (t-1)\det P_0$.

Module theory says that $\det P_*$ represents the p-Alexander polynomial for K_*. Hence Theorem 3.2 and 3.3 hold.

Proof of Theorem 3.4. By [KM], $bP_{4k+2} = \mathbb{Z}_2$. In our case, the Arf invariant of a knot coincides with that of a manifold diffeomorphic to the knot. (See [Br1] for more results on bP_{4k+2}.) Put $bP_{4k+2} = \{1, g\}$, where $g^2 = 1$.

Let V be the total space of D^{2k+1} bundle over S^{2k+1} associated with the tangent bundle of S^{2k+1}. Let $\bigvee_p$ denote a plumbing (see [Br2]). Then $\partial(V \bigvee_p V)$ represents $g \in bP_{4k+2}$. Put $M = \partial V$. By [9] and [1], M is homotopy type equivalent to $S^{2k+1} \times S^{2k}$ if and only if $k = 0, 1, 3$. Hence M is not diffeomorphic to $S^{2k+1} \times S^{2k}$ in our case.

[5] proved $M \sharp \Sigma = M$. (Hence $I(M) \cap bP_{4k+2} = I(M\sharp\Sigma) \cap bP_{4k+2} = \mathbb{Z}_2$.

Corollary 3 of [Kk] proved $I(S^p \times S^q) = \{1\}$ for $p + q \geqq 5$.

By [Kk], $S^{2k+1} \times S^{2k}\sharp\Sigma$ is not diffeomorphic to $S^{2k+1} \times S^{2k}$. Hence $I(S^{2k+1} \times S^{2k}\sharp\Sigma) \cap bP_{4k+2}$=$\{1\}$.

There are four cases. Put $T = S^{2k+1} \times D^{2k+1}$. Let $\cong$ denote a diffeomorphism.

(1)$K_+ \cong \partial(V \bigvee_p V)$, $K_- \cong \partial(V \bigvee_p T)$, $K_0 \cong \partial V$.

(2)$K_+ \cong \partial(T \bigvee_p V)$, $K_- \cong \partial(T \bigvee_p T)$ $K_0 \cong \partial T$.

(3)$K_+ \cong \partial\{(V \bigvee_p V)\natural(V \bigvee_p V)\}$, $K_- \cong \partial\{(V \bigvee_p T)\natural(V \bigvee_p V)\}$, $K_0 \cong \partial\{V\natural(V \bigvee_p V)\}$.

(4)$K_+ \cong \partial\{(T \bigvee_p V)\natural(V \bigvee_p V)\}$, $K_- \cong \partial\{(T \bigvee_p T)\natural(V \bigvee_p V)\}$, $K_0 \cong \partial\{T\natural(V \bigvee_p V))$.

The formula in Theorem 3.4 holds in each case by the above discussions. Hence the formula holds.

4. More results in the 2-knot case

Our main results can be extended to some other cases where K_+ (resp. K_-) is not a knot. In this section we show more results in the case of 2-dimensional sbmanifold case.

Theorem 4.1. *Let $\Sigma_1, ..., \Sigma_\alpha$ be connected closed oriented surfaces. Let g_i be the genus of Σ_i. Put $\beta = \Sigma_{i=1}^{\alpha} g_i$. Let K_+ (resp. K_-) be a 2-dimensional submanifold $\subset S^4$ which is diffeomorphic to a disjoint ordered oriented manifold $(\Sigma_1, ..., \Sigma_\alpha)$. Suppose $\alpha = \beta + 1$. Let K_0 be a 2-dimensional submanifold $\subset S^4$. Suppose that (K_+, K_-, K_0) is related by the $(1, 2)$-move. Then we have:*

$$\Delta_{K_+}(t) - \Delta_{K_-}(t) = (t - 1) \cdot \Delta_{K_0}(t).$$

Note. (1) If $\alpha = 1$, K_+ and K_- are S^2-knots. Then K_0 is diffeomorphic to S^2 or $S^2 \amalg T^2$. In each case Theorem 1 holds. In general, if we put $H_0(K_0; \mathbb{Q}) \cong \mathbb{Q}^{\alpha'}$ and $H_1(K_0; \mathbb{Q}) \cong \mathbb{Q}^{2\beta'}$, then $\alpha' = \beta' + 1$.

(2) Since (K_+, K_-, K_0) is related by the $(1, 2)$-move, there is a 4-ball B trivially embedded in S^4 with the following properties. We regard B as (2-disc)$\times[0, 1] \times \{t| - 1 \leqq t \leqq 1\}$.

(i) $K_+ - B$, $K_- - B$, and $K_- - B$ coincide each other.

(ii) $B \cap K_+$, $B \cap K_-$, $B \cap K_-$ are drawn as in Figure 3.

In Figure 3 we draw $B_{-0.5}\cap K_*$, $B_0\cap K_*$, $B_{0.5}\cap K_*$, where B_{t_0}=(2-disc)$\times[0,1]\times\{t|t=t_0\}$. We suppose that each vector $\overrightarrow{x}$, $\overrightarrow{y}$ in Figure 3 is a tangent vector of each disc at a point. (Note we use $\overrightarrow{x}$ (resp. $\overrightarrow{y}$) for different vectors.) The orientation of each disc in Figure 3 is determined by the each set $\{\overrightarrow{x}, \overrightarrow{y}\}$.
In [18] the author calls the operation to change K_+ into K_- (1,2)-pass-move. Around Figure 4.1 and 4.2 in [18], we wrote more explanation of the figure of $B\cap K_+$ and that of $B\cap K_-$.
(4) After sending these results (without Appendix) to several people, the author was informed Giller's article, P.627,628 of [Gi]. Only in the $n=2$ case Giller proved a result which is weaker than ours. See the Appendix for detail.

[Figure 3]

Proof of Theorem 4.1. Since $\beta=\Sigma_{i=1}^{\alpha}g_i$, all Seifert surfaces V_* for K_* has a property that $H_1(V_*;\mathbb{Q})\cong H_2(V_*;\mathbb{Q})$ and that $H_0(V_*;\mathbb{Q})\cong\mathbb{Q}$.
The left of the proof is same as the proof of Theorem 3.2.

On the condition $\alpha=\beta+1$ in Theorem 4.1 we have:

Proposition 4.2. *We CANNOT remove the condition $\alpha=\beta+1$ in Theorem 4.1 in general.*

Proof of Theorem 4.2. Let K_-, K_+, K_0 be 2-dimensional oriented subamanifold $\subset S^4$ which are diffeomorphic to T^2-knots. Suppose (K_-,K_+,K_0) is related by the $(1,2)$-move.

Suppose that

(1) K_+ bounds $V_+\cong S^1\times B^2\natural(S^2\times S^1-B^3)\natural(S^2\times S^1-B^3)$
(2) K_- bounds $V_-\cong V_+$.
(3) K_0 bounds $V_0\cong S^1\times B^2\natural(S^2\times S^1-B^3)$.

Consider the exact sequence as in Proof of Theorem 3.2:
$H_*(\amalg_{-\infty}^{\infty}V_*)\xrightarrow{f_*}H_*(\amalg_{-\infty}^{\infty}Y_*)\to H_p(\widetilde{X}_*;\mathbb{Z})$.

We can suppose that

(1) f_+ is represented by $\begin{pmatrix}3t-2 & 0 & 0\\ t-1 & 2t-1 & 0\end{pmatrix}$.
(2) f_- is represented by $\begin{pmatrix}3t-2 & 0 & t-1\\ t-1 & 2t-1 & 0\end{pmatrix}$.
(3) f_0 is represented by $\begin{pmatrix}t-1 & 2t-1\end{pmatrix}$.

The above exact sequences are:

$$\oplus^3\mathbb{Z}[t,t^{-1}]\xrightarrow[\begin{pmatrix}3t-2 & 0 & 0\\ t-1 & 2t-1 & 0\end{pmatrix}]{}\oplus^2\mathbb{Z}[t,t^{-1}]\longrightarrow\mathbb{Z}[t,t^{-1}]/\{(3t-2)\}\oplus\mathbb{Z}[t,t^{-1}]/\{(2t-1)\}\longrightarrow 0$$

$$\oplus^3\mathbb{Z}[t,t^{-1}] \xrightarrow[\begin{pmatrix} 3t-2 & 0 & t-1 \\ t-1 & 2t-1 & 0 \end{pmatrix}]{} \oplus^2\mathbb{Z}[t,t^{-1}] \longrightarrow 0 \longrightarrow 0$$

$$\oplus^2\mathbb{Z}[t,t^{-1}] \xrightarrow[\begin{pmatrix} t-1 & 2t-1 \end{pmatrix}]{} \mathbb{Z}[t,t^{-1}] \longrightarrow 0 \longrightarrow 0$$

Therefore we have:
(1) $(3t-2)(t-1)$ represents the Alexander polynomial of K_+.
(2) 1 represents the Alexander polynomial of K_-.
(3) 1 represents the Alexander polynomial of K_0.

For the above K_-, K_+, K_0, $\Delta_{K_+} - \Delta_{K_-} = (t-1)\cdot\Delta_{K_0}$ DOES NOT hold for any set of polynomials Δ_{K_-}, Δ_{K_+}, Δ_{K_0} such that Δ_{K_-} (resp. Δ_{K_+}, Δ_{K_0}) represents the Alexander polynomials of K_- (resp. K_+, K_0). The proof is completed.

Next we discuss "normalization" of the Alexander polynomials. Recall that, in the case of 1-links, we can choose a unique polynomial from the all polynomilas whose ballanced classes are the Alexander polynomial. (See e.g. [11] for detail.) However, we have:

Proposition 4.3. *In the $n \neq 1$ case in Theorem 3.2, we CANNOT choose a unique polynomial from all polynomials which represent the Alexander polynomial to be compatible with our local move formula.*

We can suppose that K_+, K_-, K_0 are trivial knots and that (K_+, K_-, K_0) can be related by the $(1,2)$-move. Becase: Let $V_+ \cong \overline{S^2 \times S^1 - B^3}$. Let $V_- \cong \overline{S^2 \times S^1 - B^3}$. Let $V_0 \cong \overline{B^3}$. Use these V_+, V_-, V_0.

If we can take a unique polynomial to represent the Alexander polynomial, then we can let the Alexander polynomial $a \cdot t^m$ for K_+, K_- and K_0, where a is a nonzero rational number. Hence $a\cdot t^m - a \cdot t^m = (t-1)\cdot a \cdot t^m$. Hence $0 = (t-1)\cdot a\cdot t^m$. It is the contardiction. Hence we CANNOT choose a unique polynomial.

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Appendix.

We explain the fact in Theorem 4.1(4) a little more.

Replace Figure 3 with Figure 3 in the definition that (K_+, K_-, K_0) is related by the $(1,2)$-move. Then we say that (K_+, K_-, K_0) is *related by the ribbon-move.*

Note. In [18] the author call the operation to change K_+ into K_- (resp. K_- into K_+) in Figure 4, ribbon-move.

By using Theorem 4.1 in this paper and Proposition 4.2, 4.3, 4.4 in [18], we have: If (K_+, K_-, K_0) is related by the ribbon-move, then (K_+, K_-, K_0) is related by the (1,2)-move.

Thus we have:

Theorem. *In Theorem 4.1, we can replace the word 'the (1,2)-move' with 'the ribbon-move.'*

Note. In P.627, 628 of [Gi], Giller proved a weaker case of this Theorem: [Gi] does not prove the case where K_+ is a sphere, K_- is a sphere, and K_0 is not a sphere. It means that [Gi]'s formula is not a local move formula in the meaning of Note 3.1. Furthermore [Gi] does not prove more than one of K_+, K_-, K_0 is a sphere. In the meaning of the following Proposition, ours are stronger than the formula in [Gi].

[Figure 4]

Note. The orientation of the part of $B \cap K_0$ derived from $B \cap K_+$ (resp. $B \cap K_+$) is given by using $B \cap K_+$ (resp. $B \cap K_+$). The orientation of $B \cap K_0$ is compatible with the part of $B \cap K_0$ derived from $B \cap K_+$ (resp. $B \cap K_+$).

It is natural to ask the following. If (K_+, K_-, K_0) is related by the $(1, 2)$-move, then do they compose a triple of Figure 1.1?

The answer is negative in general by the following Proposition.

Let $K = (K_1, K_2, K_3)$ and $K' = (K'_1, K'_2, K'_3)$ be 2-dimensional submanifolds $\subset S^4$ such that $K_1 \cong K'_1 \cong S^2$, $K_2 \cong K'_2 \cong S^2$, and $K_3 \cong K'_3 \cong \Sigma_2$, Σ_2 is the oriented closed surface with the genus two.

Supoose that $\mathrm{alk}(K_1, K_3)$ is one, $\mathrm{alk}(K_2, K_3)$ is one, where $\mathrm{alk}(\quad)$ denotes the alinking number (in [21]).

Suppose that K is changes into K' by one (1,2)-pass-move (see [18]). We can suppose this (1,2)-pass-move let $\mathrm{alk}(K_1, K_3)$ zero and let $\mathrm{alk}(K_2, K_3)$ zero.

Then we prove:

Proposition. *Let K and K' be as above. K does not change into K' by one ribbon-move.*

Proof. One ribbon-move cannot change $\mathrm{alk}(K_1, K_3)$ and $\mathrm{alk}(K_2, K_3)$ together.

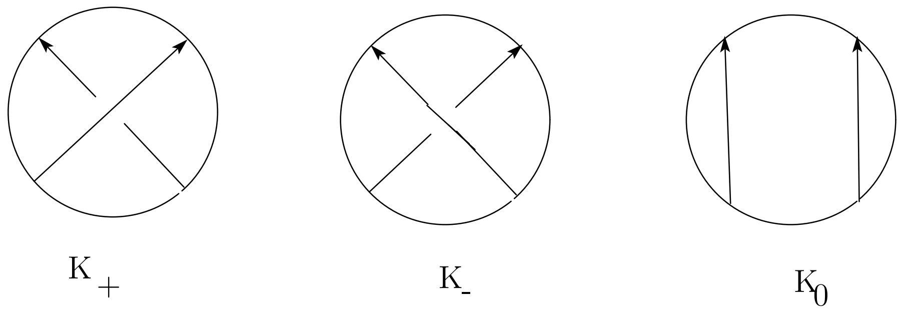

Figure 1.

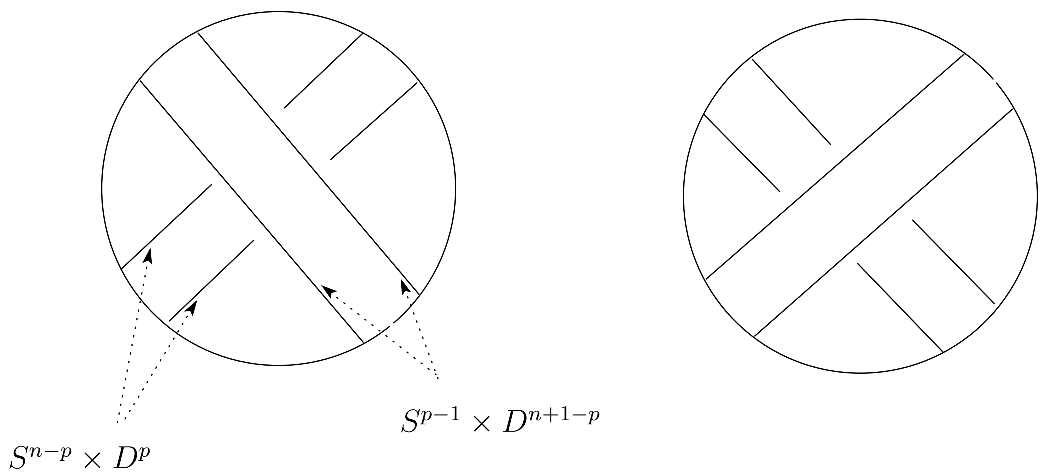

Figure 2.

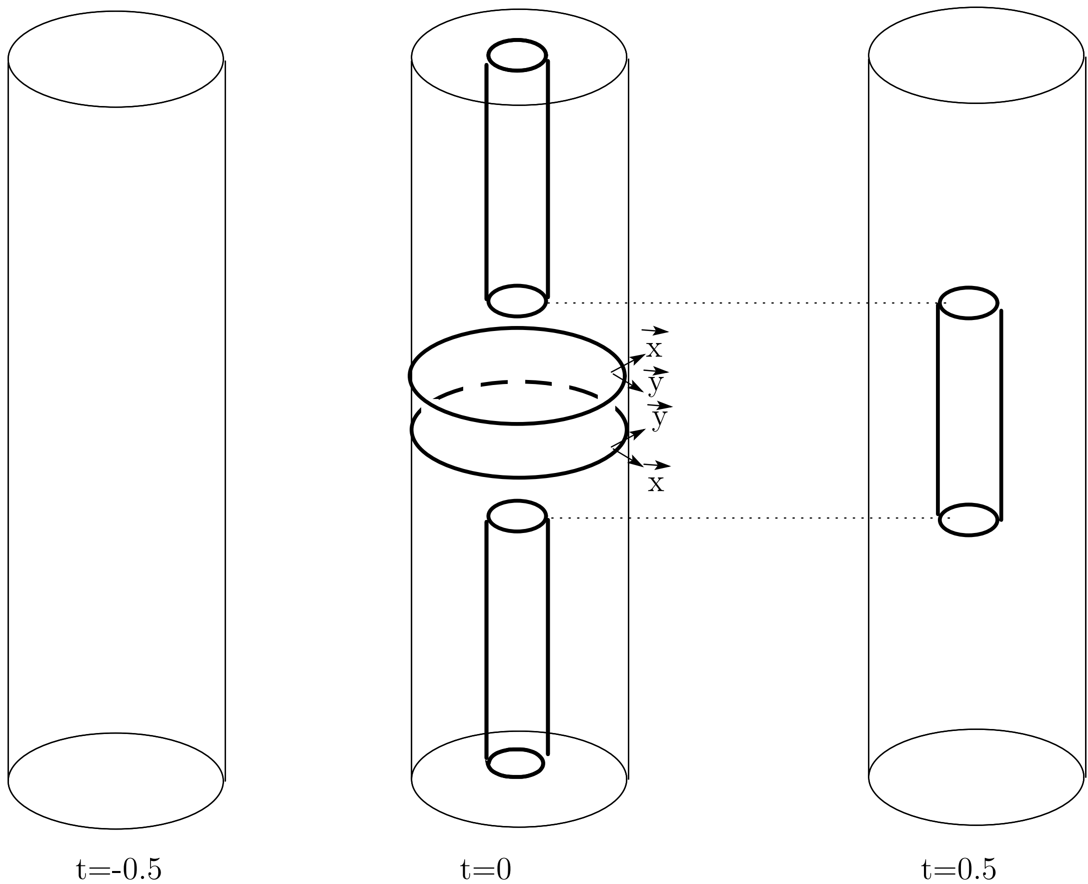

Figure 3. K_+

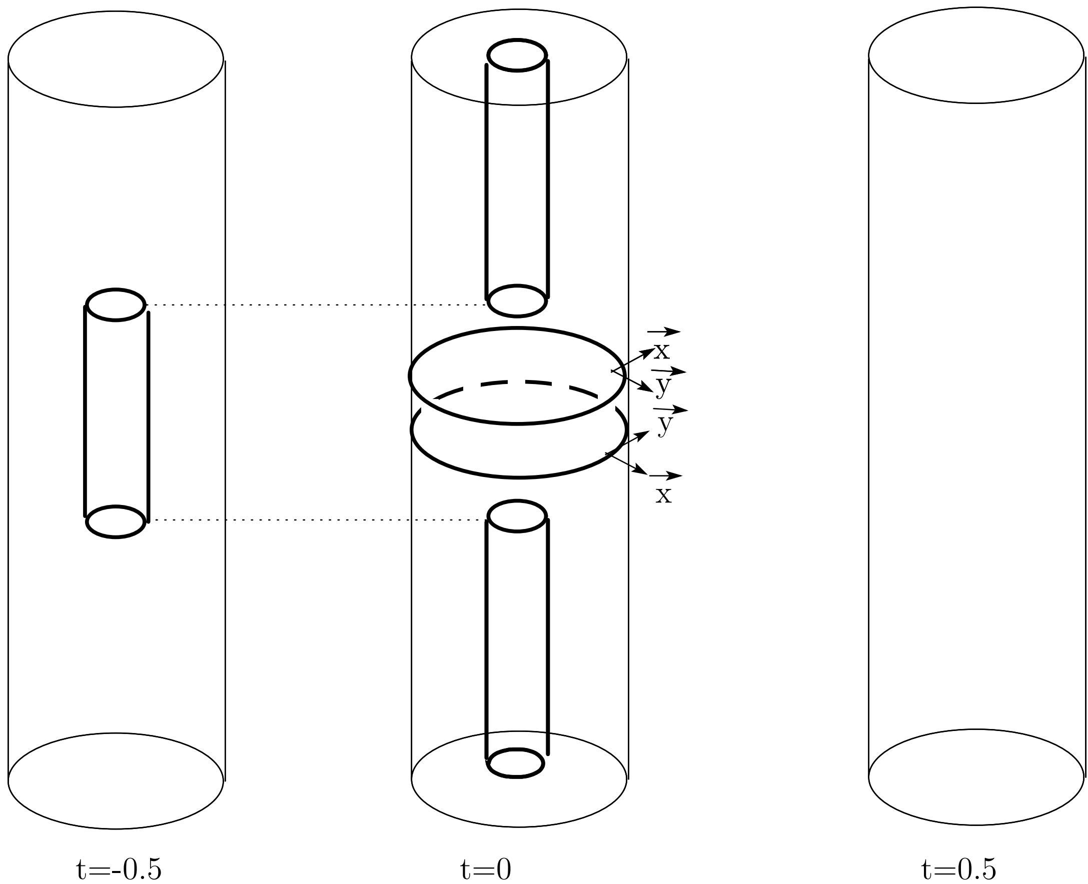

Figure 3. K_-

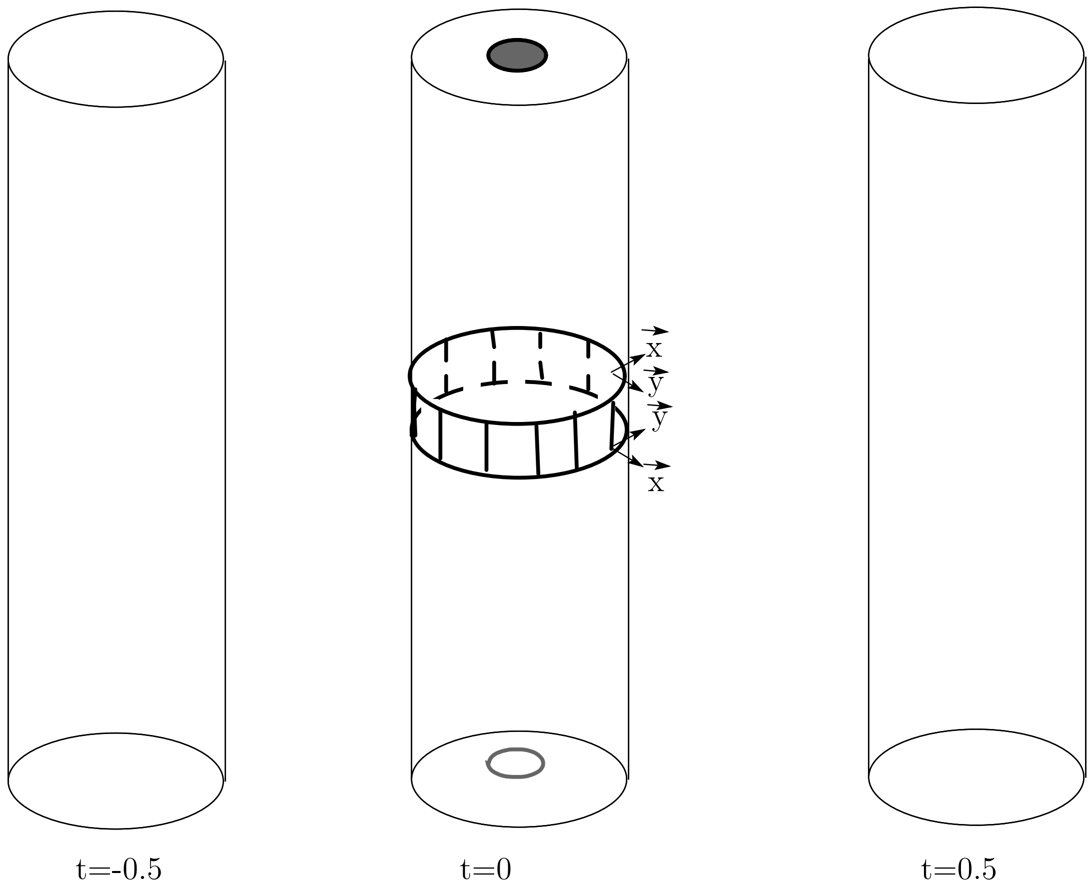

Figure 3. K_0

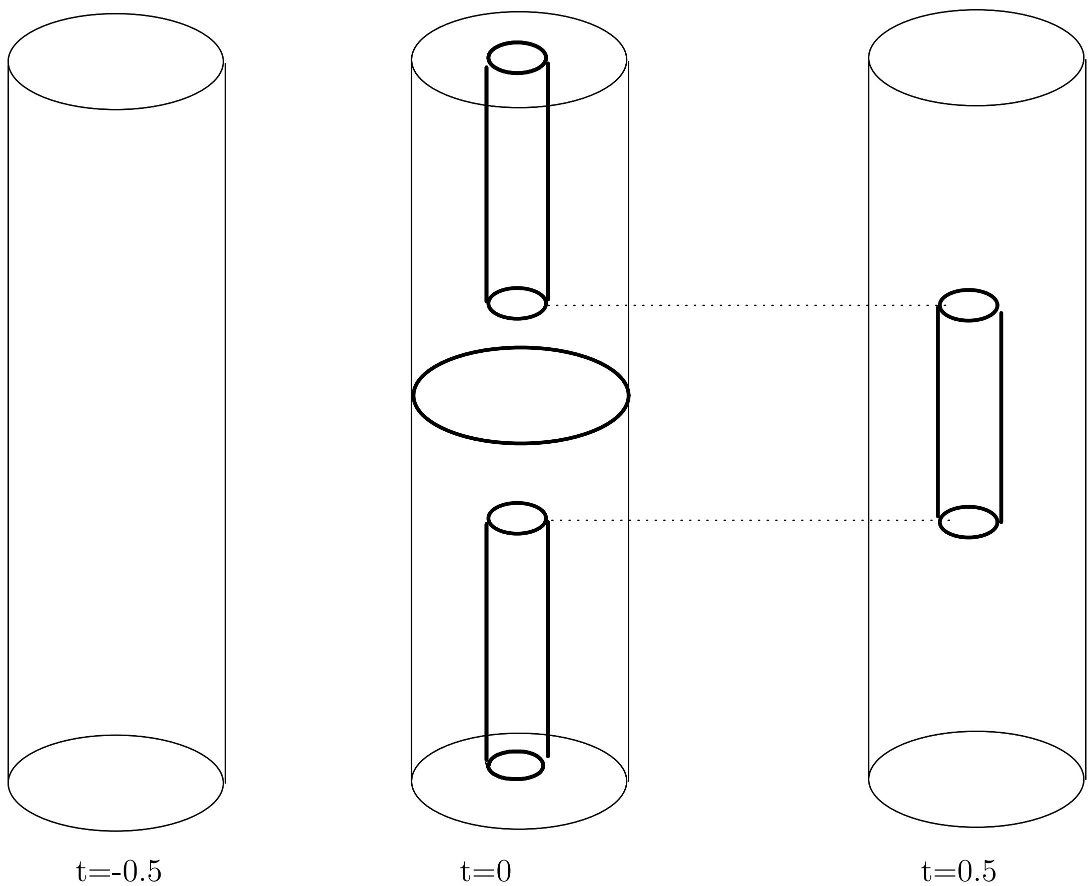

Figure 4. K_+

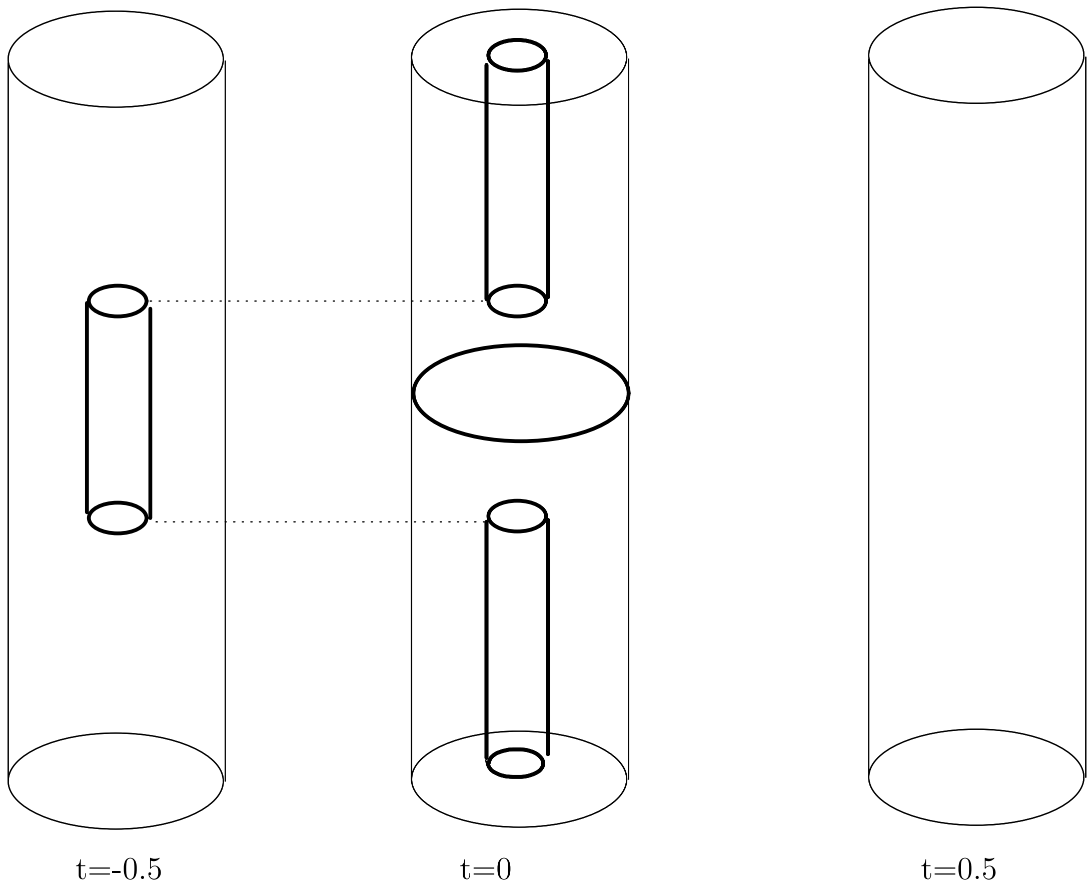

Figure 4. K_{-}

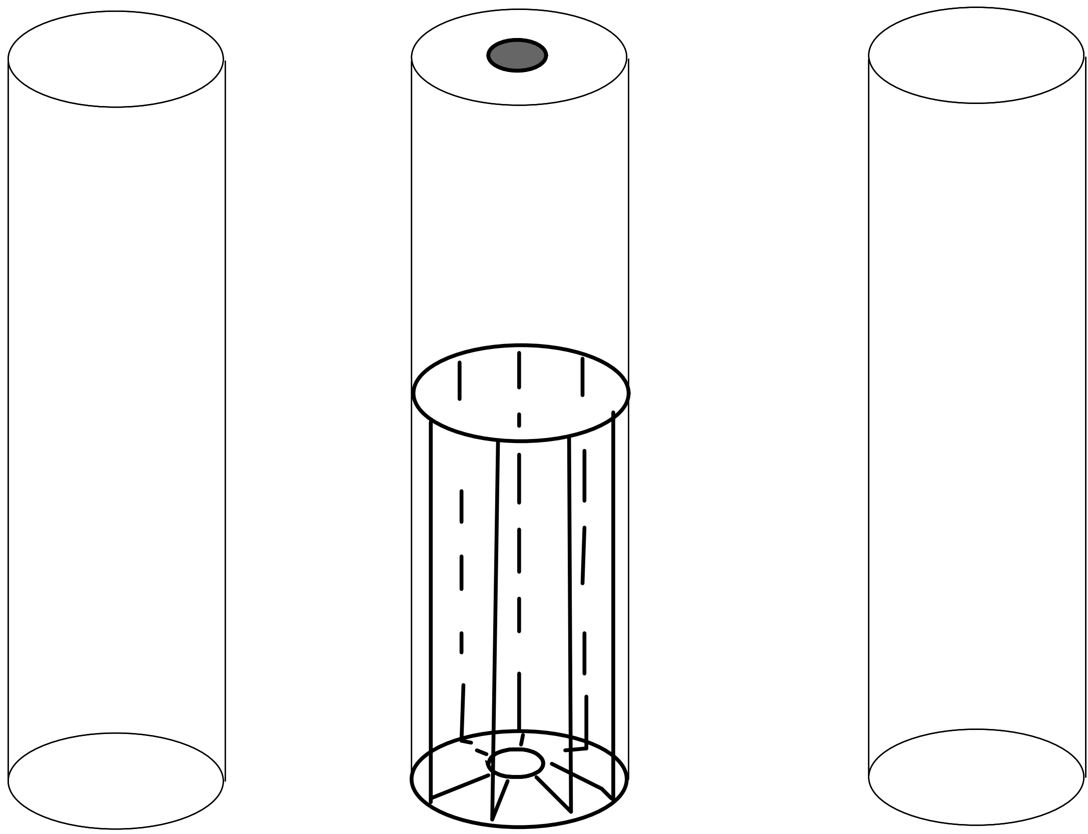

Figure 1.1

Figure 4. K_0